

# FACTORIZATION IN $SL_n(R)$ WITH ELEMENTARY MATRICES WHEN $R$ IS THE DISK ALGEBRA AND THE WIENER ALGEBRA

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**ABSTRACT.** Let  $R$  be the polydisc algebra or the Wiener algebra. It is shown that the group  $SL_n(R)$  is generated by the subgroup of elementary matrices with all diagonal entries 1 and at most one nonzero off-diagonal entry. The result an easy consequence of the deep result due to Ivarsson and Kutzschebauch [4].

## 1. INTRODUCTION

Let  $R$  be a commutative unital ring. Let  $I_n$  denote the  $n \times n$  identity matrix, that is the square matrix with all diagonal entries equal to  $1 \in R$  and off-diagonal entries equal to  $0 \in R$ . Recall that an elementary matrix  $E_{ij}(\alpha)$  over  $R$  is a matrix of the form  $I_n + \alpha \mathbf{e}_{ij}$ , where  $i \neq j$ ,  $\alpha \in R$ , and  $\mathbf{e}_{ij}$  is the  $n \times n$  matrix whose entry in the  $i$ th row and  $j$ th column is 1 and all other entries are zeros. Let  $SL_n(R)$  be the group of all  $n \times n$  matrices whose entries are elements of  $R$  and whose determinant is 1. Let  $E_n(R)$  be the subgroup of  $SL_n(R)$  generated by the elementary matrices.

A classical question in commutative algebra is the following:

**Question 1.1.** Is  $SL_n(R)$  equal to  $E_n(R)$ ?

The answer to this question depends on the ring  $R$ , and here is a list of a few known results.

- (1) If  $R = \mathbb{C}$ , then the answer is “Yes”, and this is standard exercise in linear algebra; see for example [1, Exercise 18.(c), page 71].
- (2) Let  $R$  be the polynomial ring  $\mathbb{C}[z_1, \dots, z_n]$  in the indeterminates  $z_1, \dots, z_n$  with complex coefficients.

If  $n = 1$ , then the answer is “Yes”, and this follows from the Euclidean Division Algorithm in  $\mathbb{C}[z]$ .

If  $n = 2$ , then the answer is “No”, and [2] gave the following counterexample:

$$\begin{bmatrix} 1 + z_1 z_2 & z_1^2 \\ -z_2^2 & 1 - z_1 z_2 \end{bmatrix} \in SL_2(\mathbb{C}[z_1, z_2]) \setminus E_2(\mathbb{C}[z_1, z_2]).$$

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For  $n \geq 3$ , the answer is “Yes”, and this is the  $K_1$ -analogue of Serre’s Conjecture, which is the Suslin Stability Theorem [5].

- (3) The case of  $R$  being a ring of continuous functions was considered in [6]. Let  $C(X; \mathbb{C})$  be the ring of continuous complex-valued functions on the finite-dimensional normal topological space  $X$  with pointwise operations.  $C_b(X; \mathbb{C})$  denotes the subring of  $C(X; \mathbb{C})$  consisting of *bounded* functions. It was shown in [6] that for  $R = C(X; \mathbb{C})$  or  $C_b(X; \mathbb{C})$ , the answer is “Yes” if there is no homotopy obstruction. Indeed, if  $E$  is an elementary matrix, then  $(X \ni) x \mapsto E(x) \in SL_n(\mathbb{C})$  is null-homotopic (to the constant map  $x \mapsto I_n : X \rightarrow SL_n(\mathbb{C})$ ). So it follows that if  $\pi(F)$  denotes the homotopy class of the map  $x \mapsto F(x) : X \rightarrow SL_n(\mathbb{C})$  corresponding to  $F \in SL_n(R)$ , then a necessary condition for  $F \in E_n(C(X; \mathbb{C}))$  is that  $\pi(F) = 0$ . It turns out that this condition is also sufficient, and this is the content of [6, Theorem 4].
- (4) Based on the above result, it is natural to consider the question also for the ring  $\mathcal{O}(X)$  of holomorphic functions on Stein spaces in  $\mathbb{C}^n$ . This was posed as an explicit open problem by Gromov in [3], and was recently solved by Ivarsson and Kutzschebauch [4]. The main result in [4] is the following:

**Theorem 1.2** ([4]). *If  $X$  is a finite-dimensional reduced Stein space and  $F : X \rightarrow SL_n(\mathbb{C})$  is a holomorphic mapping that is null-homotopic, then there exists a natural number  $K$  and holomorphic mappings  $G_1, \dots, G_K : X \rightarrow \mathbb{C}^{m(m-1)/2}$  such that  $F$  can be written as a product of upper and lower diagonal unipotent matrices*

$$F(x) = M_1(G_1(x)) \cdots M_K(G_K(x)), \quad x \in X,$$

where the matrices  $M_j(G_j(x))$  are defined by

$$M_j(G_j(x)) := \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ G_j(x) & & 1 \end{bmatrix} \quad \text{if } j \text{ is odd,}$$

while

$$M_j(G_j(x)) := \begin{bmatrix} 1 & & G_j(x) \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \quad \text{if } j \text{ is even.}$$

In particular, the assumption of null-homotopy is always satisfied if  $X$  is contractible.

We wish to consider Question 1.1 for commutative, semisimple, unital complex Banach algebras  $R$ . A special case is when  $R = C_b(X; \mathbb{R})$ , where  $X$  is a compact Hausdorff topological space, and item (3) above describes the answer in this special case. Motivated by this, we formulate the following question/conjecture, but first we introduce some convenient notation.

Let  $R$  be a commutative, semisimple, unital complex Banach algebra with maximal ideal space denoted by  $X_R$ , equipped with the weak-\* topology induced from the dual space  $R^* := \mathcal{L}(R; \mathbb{C})$  of  $R$ .

Let  $\hat{\cdot} : R \rightarrow C(X_R; \mathbb{C})$  denote the Gelfand transform. For  $F \in SL_n(R)$ , let  $\hat{F}$  be the matrix with elements in  $C(X_R; \mathbb{C})$  obtained by taking the Gelfand transform of the entries of  $F$ , and  $\pi(\hat{F})$  denotes the homotopy class of  $\varphi \mapsto \hat{F}(\varphi) : X_R \rightarrow SL_n(\mathbb{C})$ .

**Conjecture 1.3.** Let  $R$  be a commutative, semisimple, unital complex Banach algebra.  $F \in SL_n(R)$  belongs to  $E_n(R)$  if and only if  $\pi(\hat{F}) = 0$ .

We consider Question 1.1 for two important Banach algebras of holomorphic functions: the polydisc algebra  $A(\overline{\mathbb{D}}^n)$  and the Wiener algebra  $W^+(\overline{\mathbb{D}}^n)$ .

Let  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  and  $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ . Let  $d \in \mathbb{N}$ . The Wiener algebra  $W^+(\overline{\mathbb{D}}^n)$  is the Banach algebra defined by

$$W^+(\overline{\mathbb{D}}^d) = \left\{ \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} a_{(k_1, \dots, k_d)} z_1^{k_1} \cdots z_d^{k_d} : \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} |a_{(k_1, \dots, k_d)}| < \infty \right\},$$

with pointwise addition and multiplication, and the  $\|\cdot\|_1$ -norm given by

$$\|f\|_1 = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} |a_{(k_1, \dots, k_d)}|, \quad f = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} a_{(k_1, \dots, k_d)} z_1^{k_1} \cdots z_d^{k_d}.$$

The polydisc algebra  $A(\overline{\mathbb{D}}^d)$  is the Banach algebra of all continuous functions  $f : \overline{\mathbb{D}}^d \rightarrow \mathbb{C}$  which are holomorphic in  $\mathbb{D}^d$ , with pointwise addition and multiplication, and the supremum norm  $\|\cdot\|_{\infty}$  given by

$$\|f\|_{\infty} := \sup_{(z_1, \dots, z_d) \in \overline{\mathbb{D}}^d} |f(z_1, \dots, z_d)|, \quad f \in A(\overline{\mathbb{D}}^d).$$

The ball algebra  $A(\overline{\mathbb{B}}_d)$  is defined similarly, with the polydisc  $\overline{\mathbb{D}}^d$  replaced by the ball

$$\overline{\mathbb{B}}_d := \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 \leq 1\}.$$

For a  $n \times n$  matrix  $F$  with entries in  $A(\overline{\mathbb{D}}^d)$ ,  $A(\overline{\mathbb{B}}_d)$  or  $W^+(\overline{\mathbb{D}}^d)$ , we define

$$\|F\| := \sum_{i,j=1}^n \|F_{ij}\|_{\infty},$$

where  $F_{ij}$  denotes the entry in the  $i$ th row and  $j$ th column of  $F$ . Then  $\|FG\| \leq \|F\|\|G\|$ , for  $n \times n$  matrices  $F, G$  with entries from any of the Banach algebras  $A(\overline{\mathbb{D}}^d)$ ,  $A(\overline{\mathbb{B}}_d)$  or  $W^+(\overline{\mathbb{D}}^d)$ .

Our main result is the following.

**Theorem 1.4.** If  $R = A(\overline{\mathbb{D}}^d)$ ,  $A(\overline{\mathbb{B}}_d)$  or  $W^+(\overline{\mathbb{D}}^d)$ , then  $SL_n(R) = E_n(R)$ .

If  $R = A(\overline{\mathbb{D}}^d)$  or  $W^+(\overline{\mathbb{D}}^d)$ , then in both cases, the maximal ideal space  $X_R$  can be identified with  $\overline{\mathbb{D}}^d$  as a topological space. Similarly,  $X_{A(\overline{\mathbb{B}}_d)} = \overline{\mathbb{B}}_d$ . If Conjecture 1.3 is true, then Theorem 1.4 follows from the observation that  $\overline{\mathbb{D}}^d, \overline{\mathbb{B}}_d$  are contractible (since then  $\pi(\widehat{F})$  is always trivial).

We will derive our main result as a consequence of the result from [4] quoted above, and [6, Lemma 9] reproduced below.

**Lemma 1.5** ([6]). *Let  $R$  be a commutative topological unital ring such that the set of invertible elements of  $R$  is open in  $R$ . If  $F \in SL_n(R)$  is sufficiently close to  $I_n$ , then  $F$  belongs to  $E_n(R)$ .*

## 2. PROOF OF THEOREM 1.4

*Proof.* We will simply prove the result in the case of the disc algebra  $A(\overline{\mathbb{D}}^d)$ ; the proofs in the cases of the ball algebra  $A(\overline{\mathbb{B}}_d)$  and the Wiener algebra being analogous.

Let  $F \in SL_n(A(\overline{\mathbb{D}}^d))$ . Let  $r \in (0, 1)$  (to be determined later). Define

$$F_r(z_1, \dots, z_d) := F(rz_1, \dots, rz_d), \quad (z_1, \dots, z_d) \in \mathbb{D}^d.$$

As  $F_r \in \mathcal{O}(\frac{1}{r}\mathbb{D}^d)$ , and  $\det F_r \equiv 1$ , it follows from Theorem 1.2 (since  $\frac{1}{r}\mathbb{D}^d$  is a contractible Stein domain) that there are elementary matrices  $G_1, \dots, G_K$  belonging to  $E_n(\mathcal{O}(\frac{1}{r}\mathbb{D}^d))$  such that

$$F_r = E_1 \cdots E_K \in E_n(\mathcal{O}(\frac{1}{r}\mathbb{D}^d)) \subset E_n(A(\overline{\mathbb{D}}^d)).$$

Thus  $F(I_n + F^{-1}(F_r - F)) = F_r \in E_n(A(\overline{\mathbb{D}}^d))$ . As  $\det F = \det F_r = 1$ , it follows that also  $\det(I_n + F^{-1}(F_r - F)) = 1$ . We will be done if we manage to show that  $I_n + F^{-1}(F_r - F) \in E_n(A(\overline{\mathbb{D}}^d))$  too. But this is clear by Lemma 1.5, since

$$\left\| (I_n + F^{-1}(F_r - F)) - I_n \right\| = \|F^{-1}(F_r - F)\| \leq \|F^{-1}\| \|F_r - F\|,$$

and we can make  $\|F_r - F\|$  as small as we like by choosing  $r$  close enough to 1.  $\square$

**Remark 2.1.** The above proof also works for some other Banach algebras of smooth functions contained in the polydisc algebra, for example, if  $N \in \mathbb{N}$ , the Banach algebra  $\partial^{-N}A(\overline{\mathbb{D}}^d)$  of all functions  $f \in A(\overline{\mathbb{D}}^d)$  whose complex partial derivatives of all orders up to  $N$  belong to  $A(\overline{\mathbb{D}}^d)$ , with the norm

$$\|f\|_{\partial^{-N}A(\overline{\mathbb{D}}^d)} := \sum_{\alpha_1 + \dots + \alpha_d \leq N} \frac{1}{\alpha_1! \cdots \alpha_d!} \sup_{(z_1, \dots, z_d) \in \mathbb{D}^d} \left| \frac{\partial^{\alpha_1 + \dots + \alpha_d} f}{\partial z_1^{\alpha_1} \cdots \partial z_d^{\alpha_d}}(z_1, \dots, z_d) \right|.$$

In light of Theorem 1.4, it is natural to ask the analogous question also for the Hardy algebra. Recall that if  $U$  is an open set in  $\mathbb{C}^d$ , then the Hardy algebra  $H^\infty(U)$  is the Banach algebra of all complex-valued functions on  $U$  that are bounded and holomorphic in  $U$ .

**Conjecture 2.2.**  $SL_n(H^\infty(U)) = E_n(H^\infty(U))$  if  $U$  is the polydisc  $\mathbb{D}^d$  or open unit ball  $U = \mathbb{B}_d := \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_1|^2 + \dots + |z_d|^2 < 1\}$ .

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